



TITLE:

Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations (Modeling and Complex analysis for functional equations)

AUTHOR(S):

Kamo, Ken-ichi; Usami, Hiroyuki

CITATION:

Kamo, Ken-ichi ...[et al]. Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations (Modeling and Complex analysis for functional equations). 数理解析研究所講究録 2008, 1582: 43-52

ISSUE DATE:

2008-02

URL:

<http://hdl.handle.net/2433/81456>

RIGHT:

Asymptotic forms of slowly decaying positive solutions of second-order quasilinear ordinary differential equations 準線型 2 階常微分方程式の緩減衰正值解の漸近形 について

札幌医科大学・医・加茂憲一 (Ken-ichi Kamo)
Sapporo Medical University
広島大学・理・宇佐美広介 (Hiroyuki Usami)
Hiroshima University

1 Introduction

Let us consider the quasilinear ODE

$$(a(t)|u'|^{\alpha-1}u')' + b(t)|u|^{\lambda-1}u = 0, \quad \text{near } +\infty \quad (\text{A})$$

where we assume that $\alpha > 0$ and $\lambda > 0$ are constants, $a(t)$ and $b(t)$ are positive continuous functions satisfying $\int_t^\infty a(s)^{-1/\alpha} ds < \infty$. Every positive solution u of (A) satisfies one of the following three asymptotic properties as $t \rightarrow \infty$:

$$u(t) \sim c_1 \quad \text{for some constant } c_1 > 0; \quad (1.1)$$

$$u(t) \sim c_2 \int_t^\infty a(s)^{-1/\alpha} ds \quad \text{for some constant } c_2 > 0; \quad (1.2)$$

and

$$u(t) \rightarrow 0 \quad \text{and} \quad \frac{u(t)}{\int_t^\infty a(s)^{-1/\alpha} ds} \rightarrow \infty. \quad (1.3)$$

Asymptotic properties of solutions u satisfying either (1.1) or (1.2) were widely investigated. For example, necessary and sufficient conditions of existence of such solutions were established in [4, 7]. On the other hand there seems to be less information about qualitative properties of solutions u satisfying (1.3). Motivated by this fact, in the article we will discuss about asymptotic behavior of solutions u satisfying (1.3); in particular, we try to find exact asymptotic forms of such solutions near $+\infty$. In what follows we refer solutions u satisfying (1.3) as *slowly decaying solutions*.

Remark 1. When $\int_t^\infty a(s)^{-1/\alpha} ds = \infty$, Eq (A) reduces to the simpler one of the form

$$(|u'|^{\alpha-1}u')' + \tilde{b}(t)|u|^{\lambda-1}u = 0 \quad \text{near } +\infty,$$

where $\tilde{b}(t)$ is a positive continuous function. Studies of this equation were, for example, the main objective of [6]; and asymptotic properties of solutions have been fully established

2 Preparatory observations and results

Asymptotic forms of slowly decaying solutions may be strongly affected by those of coefficient functions $a(t)$, $b(t)$ and the exponents α and λ . Therefore let us consider the following ODE, which has more restrictive appearance than Eq (A):

$$(t^\beta |u'|^{\alpha-1} u')' + t^\sigma (1 + \varepsilon(t)) |u|^{\lambda-1} u = 0 \quad \text{near } +\infty. \quad (\text{E})$$

In the sequel we assume the next conditions:

(A₁) α, β, λ and σ are constants satisfying $\lambda > \alpha > 0$ and $\beta > \alpha$;

(A₂) $\varepsilon(t)$ is a continuous (or C^1 -)function defined near $+\infty$ satisfying $\lim_{t \rightarrow \infty} \varepsilon(t) = 0$.

Additional conditions will be given later.

Since we can regard Eq (E) as a "perturbed equation" of the ODE

$$(t^\beta |u'|^{\alpha-1} u')' + t^\sigma |u|^{\lambda-1} u = 0 \quad \text{near } +\infty, \quad (\text{E}_0)$$

we conjecture that slowly decaying solutions of Eq (E) and those of Eq (E₀) may have the same asymptotic behavior near $+\infty$ in some sense, if $\varepsilon(t)$ is sufficiently small. It is easily seen that Eq (E₀) has an exact slowly decaying solution u_0 given by

$$u_0(t) = \hat{C} t^{-k}, \quad (2.1)$$

where

$$k = \frac{1 + \sigma - (\beta - \alpha)}{\lambda - \alpha}, \quad \text{and} \quad \hat{C}^{\lambda-\alpha} = k^\alpha \{\beta - \alpha(k+1)\}$$

if

$$(\beta - \alpha) - 1 < \sigma < \frac{\lambda}{\alpha}(\beta - \alpha) - 1. \quad (2.2)$$

Below we always assume (2.2). We can show that our conjecture is true in various cases:

Theorem 1. *Let $\alpha \leq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If*

$$\text{either } \int_0^\infty \frac{\varepsilon(t)^2}{t} dt < \infty \quad \text{or} \quad \int_0^\infty |\varepsilon'(t)| dt < \infty, \quad (2.3)$$

then every slowly decaying positive solution u of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \rightarrow \infty$, where $u_0(t)$ is given by (2.1).

Theorem 2. *Let $\alpha \geq 1$ and $\beta - \alpha(k+1) - k \neq 0$. If*

$$\lim_{t \rightarrow \infty} t \varepsilon'(t) = 0 \quad \text{and} \quad \int_0^\infty |\varepsilon'(t)| dt < \infty, \quad (2.4)$$

then every slowly decaying positive solution u of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \rightarrow \infty$.

Theorem 3. Let $\alpha \geq 1$ and $\alpha(2k+1) - \beta < 0$. If (2.3) holds, then every slowly decaying positive solution u of Eq (E) satisfies $u(t) \sim u_0(t)$ as $t \rightarrow \infty$.

Example 1. Let $N > m > 1$ and $N \geq 2$. Consider radial solutions $u = u(|x|)$ of the following quasilinear PDE in an exterior domain of \mathbf{R}^N :

$$\operatorname{div}(|Du|^{m-2}Du) + |x|^\ell(1 + |x|^{-\theta})|u|^{\lambda-1}u = 0 \quad \text{near } \infty,$$

where $\lambda > m - 1$, $\ell \in \mathbf{R}$, $\theta > 0$, and $-m < \ell < \frac{\lambda}{m-1}(N - m) - N$. We know that u solves the ODE

$$(r^{N-1}|u'|^{m-2}u')' + r^{N-1+\ell}(1 + r^{-\theta})|u|^{\lambda-1}u = 0 \quad \text{near } +\infty.$$

By Theorems 1 and 2, if $\lambda \neq (mN - N + m\ell)/(N - m)$, then every slowly decaying positive solution u of this equation satisfies

$$u(r) \sim Ar^{-(\ell+m)/(\lambda-m+1)} \quad \text{as } r \rightarrow +\infty,$$

where A is a positive constant given by

$$A^{\lambda-m+1} = \left(\frac{\ell + m}{\ell - m + 1} \right)^{m-1} \cdot \frac{N\lambda - Nm + N - m\ell - m\lambda + \ell}{\lambda - m + 1}.$$

Remark 1. For the autonomous equation $\operatorname{div}(|Du|^{m-2}Du) + |u|^{\lambda-1}u = 0$, the assertion of Example 1 was established in [1] based on the theory of autonomous dynamical systems. Related results are found in [3, 5].

3 Sketches of the proof of the results

We give the outline of the proof of Theorems 1 and 2. We begin with several auxiliary results.

Lemma 1. Let $u(t)$ be a slowly decaying positive solution of (E). Then

$$u(t) = O(u_0(t)) \quad \text{and} \quad u'(t) = O(|u'_0(t)|) \quad \text{as } t \rightarrow \infty. \quad (3.1)$$

Proof. An integration of the both sides of Eq (E) on $[t_0, t]$ gives

$$t^\beta(-u'(t))^\alpha \geq \int_{t_0}^t r^\sigma(1 + \varepsilon(r))u^\lambda dr,$$

where t_0 is a sufficiently large number. Since u is a decreasing function, we have

$$t^\beta(-u'(t))^\alpha \geq u(t)^\lambda \int_{t_0}^t r^\sigma(1 + \varepsilon(r))dr;$$

that is,

$$-u'(t)u(t)^{-\lambda/\alpha} \geq \left(t^{-\beta} \int_{t_0}^t r^\sigma (1 + \varepsilon(r)) dr \right)^{1/\alpha}.$$

One more integration of the both sides gives the estimates for u in (3.1).

To get the estimates for u' , it suffices to notice the inequality

$$t^\beta (-u'(t))^\alpha \leq C_1 \int_{t_0}^t r^\sigma u(r)^\lambda dr,$$

where $C_1 > 0$ is a constant. Note that, to get this inequality, we must use the property $\lim_{t \rightarrow \infty} t^\beta (-u'(t))^\alpha = \infty$.

Lemma 2. *Let $u(t)$ be a slowly decaying positive solution of (E). Put $t = e^s$ and $u/u_0 = v$. Then*

- (i) v , and \dot{v} are bounded, and $\dot{v} - kv < 0$ near $+\infty$, where $\cdot = d/ds$;
- (ii) v satisfies the ODE

$$\{(kv - \dot{v})^\alpha\}' + \{\beta - \alpha(k+1)\}(kv - \dot{v})^\alpha - \hat{C}^{\lambda-\alpha}\{1 + \delta(s)\}v^\lambda = 0 \quad \text{near } +\infty, \quad (3.2)$$

where $\delta(s) = \varepsilon(e^s)$.

The proof of this lemma is based on direct computations; hence we omit it.

Remark 2. Equation (3.2) can be rewritten as

$$\ddot{v} + \left(\frac{\beta}{\alpha} - 2k - 1 \right) \dot{v} - k \left(\frac{\beta}{\alpha} - k - 1 \right) v + \hat{C}^{\lambda-\alpha} \{1 + \delta(s)\} v^\lambda = 0. \quad (3.3)$$

Lemma 3. *Let $f(s)$ be a C^1 -function near $+\infty$ satisfying $\dot{f}(s) = O(1)$ as $s \rightarrow \infty$ and $\int^\infty f(s)^2 ds < \infty$. Then $\lim_{s \rightarrow \infty} f(s) = 0$.*

The proof of this lemma will be found in [6].

Proof of Theorem 1. By the change of variables $(t, u) \mapsto (s, v)$ introduced in Lemma 2, we obtain Eq (3.2). We note that the integral conditions indicated in (2.3) are equivalent to

$$\int^\infty \delta(s)^2 ds < \infty \quad (3.4)$$

and

$$\int^\infty |\dot{\delta}(s)| ds < \infty, \quad (3.5)$$

respectively.

Step 1. We show that $\int^\infty \dot{v}(s)^2 ds < \infty$. We multiply Eq (3.2) by \dot{v} , and integrate the resulting equation on $[s_0, s]$ to obtain

$$\int_{s_0}^s \{(kv - \dot{v})^\alpha\}' \dot{v} dr + \{\beta - \alpha(k+1)\} \int_{s_0}^s (kv - \dot{v})^\alpha \dot{v} dr$$

$$-\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^s \delta(r)v^\lambda \dot{v} dr = \text{const.} \quad (3.6)$$

Since integral by parts implies that

$$\begin{aligned} \int_{s_0}^s \{(kv - \dot{v})^\alpha\} \dot{v} dr &= - \int_{s_0}^s \{(kv - \dot{v})^\alpha\} (kv - \dot{v}) dr + k \int_{s_0}^s \{(kv - \dot{v})^\alpha\} v dr \\ &= -\frac{\alpha}{\alpha+1} (kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha - k \int_{s_0}^s (kv - \dot{v})^\alpha \dot{v} dr + \text{const}, \end{aligned}$$

we obtain from (3.6)

$$\begin{aligned} -\frac{\alpha}{\alpha+1} (kv - \dot{v})^{\alpha+1} + kv(kv - \dot{v})^\alpha + \{\beta - \alpha(k+1) - k\} \int_{s_0}^s (kv - \dot{v})^\alpha \dot{v} dr \\ -\frac{\hat{C}^{\lambda-\alpha}}{\lambda+1}v^{\lambda+1} - \hat{C}^{\lambda-\alpha} \int_{s_0}^s \delta(r)v^\lambda \dot{v} dr = \text{const.} \end{aligned}$$

The boundedness of v and \dot{v} shown in Lemma 2 imply that

$$\{\beta - \alpha(k+1) - k\} \int_{s_0}^s (kv - \dot{v})^\alpha \dot{v} dr - \hat{C}^{\lambda-\alpha} \int_{s_0}^s \delta(r)v^\lambda \dot{v} dr = O(1) \quad \text{as } s \rightarrow \infty. \quad (3.7)$$

Now, since $0 < \alpha \leq 1$, the inequality

$$\begin{aligned} (X^\alpha - Y^\alpha)(X - Y) &\geq c_0(X - Y)^2(X + Y)^{\alpha-1} \\ &\text{for all } X, Y \geq 0 \text{ with } X + Y > 0 \end{aligned} \quad (3.8)$$

holds for some constant $c_0 > 0$. Therefore we obtain

$$\{(kv)^\alpha - (kv - \dot{v})^\alpha\} \dot{v} \geq c_0((kv) + (kv - \dot{v}))^{\alpha-1} \dot{v}^2;$$

that is,

$$(kv - \dot{v})^\alpha \dot{v} \leq -c_1 \dot{v}^2 + k^\alpha v^\alpha \dot{v}, \quad (3.9)$$

where $c_1 > 0$ is a constant. Let $\beta - \alpha(k+1) - k > 0$. From (3.7) and (3.9) we find that

$$\begin{aligned} -c_1 \{\beta - \alpha(k+1) - k\} \int_{s_0}^s \dot{v}^2 dr + \{\beta - \alpha(k+1) - k\} \frac{k^\alpha}{\alpha+1} v^{\alpha+1} \\ \geq \hat{C}^{\lambda-\alpha} \int_{s_0}^s \delta(r)v^\lambda \dot{v} dr + O(1) \quad \text{as } s \rightarrow \infty. \end{aligned} \quad (3.10)$$

Suppose $\int^\infty \varepsilon(t)^2 dt/t < \infty$, that is, (3.4) holds. Schwarz's inequality and (3.10) imply that

$$\begin{aligned} c_2 \int_{s_0}^s \dot{v}^2 dr &\leq c_3 - c_4 \int_{s_0}^s \delta(r)v^\lambda \dot{v} dr \\ &\leq c_3 + c_5 \left(\int_{s_0}^s \delta(r)^2 dr \right)^{1/2} \left(\int_{s_0}^s \dot{v}^2 dr \right)^{1/2} \end{aligned}$$

with some positive constants c_2, c_3, c_4 and c_5 . We therefore obtain $\int^\infty \dot{v}^2 dr < \infty$. Suppose next $\int^\infty |\varepsilon'(t)| dt < \infty$, that is, (3.5) holds. We find from (3.10) that

$$\begin{aligned} c_2 \int_{s_0}^s \dot{v}^2 dr &\leq c_3 - c_4 \int_{s_0}^s \delta(r) \left(\frac{v^{\lambda+1}}{\lambda+1} \right) dr \\ &\leq c_6 - \frac{c_4}{\lambda+1} \delta(s) v^{\lambda+1} - c_7 \int_{s_0}^s \dot{\delta}(r) v^{\lambda+1} dr \leq c_8 + c_9 \int_{s_0}^s |\dot{\delta}(r)| dr, \end{aligned}$$

where c_6, c_7, c_8 and c_9 are some positive constants. Hence we obtain $\int^\infty \dot{v}^2 dr < \infty$. The case where $\beta - \alpha(k+1) - k < 0$ can be treated similarly.

Since we have shown $\int^\infty \dot{v}^2 dr < \infty$, and $\alpha \leq 1$, Eq (3.3) shows that $\dot{v} = O(1)$ as $s \rightarrow \infty$. Therefore by Lemma 3 we find that $\lim_{s \rightarrow \infty} \dot{v}(s) = 0$.

Step 2. We show that $\liminf_{s \rightarrow \infty} v(s) > 0$. To see this by contradiction, we will derive a contradiction by assuming $\liminf_{s \rightarrow \infty} v(s) = 0$. The argument is divided into the two cases:

Case (a): $v(s)$ monotonically decreases to 0 (and so, $\dot{v}(s) \leq 0$);

Case (b): $\dot{v}(s)$ changes the sign in any neighborhood of $+\infty$.

Let case (a) occur. Put $v = x_1$ and $\dot{v} = x_2$, and $x = {}^t(x_1, x_2)$. Then, x satisfies the binary system

$$\dot{x} = Ax + f(s, x), \quad (3.11)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ k\left(\frac{\beta}{\alpha} - k - 1\right) & -\left(\frac{\beta}{\alpha} - 2k - 1\right) \end{pmatrix},$$

and

$$f(s, x) = \begin{pmatrix} 0 \\ -\frac{\dot{v}}{\alpha} \{1 + \delta(s)\} (k|x_1| + |x_2|)^{1-\alpha} |x_1|^\lambda \end{pmatrix}.$$

Here we have used the fact that $v(s) > 0$ and $\dot{v}(s) \leq 0$. Since

$$(k|x_1| + |x_2|)^{1-\alpha} |x_1|^\lambda \leq (\max\{1, k\})^{1-\alpha} (|x_1| + |x_2|)^{\lambda-\alpha+1},$$

and $(v(s), \dot{v}(s))$ corresponds to a solution $x(s)$ of system (3.11) satisfying $\lim_{s \rightarrow \infty} x(s) = 0$, by [2, Chapter 3, Theorem 5] we have

$$\lim_{s \rightarrow \infty} \frac{\log \|x(s)\|}{s} = \Lambda, \quad (3.12)$$

where Λ is the real part of an eigenvalue of A . All the eigenvalues of A are k and $-(\beta/\alpha - k - 1)$; the former is positive and the latter negative. Since $\|x(s)\| \rightarrow 0$, we have $\Lambda = -(\beta/\alpha - k - 1)$. By the assumption (2.2) we find a small $\eta > 0$ satisfying $\sigma + \lambda(-\beta/\alpha + 1) + \lambda\eta < -1$. By (3.12) we obtain

$$v(s) \leq e^{\{-(\beta/\alpha - k - 1) + \eta\}s} \quad \text{near } +\infty.$$

This means that $u(t) \leq t^{-\beta/\alpha+1+\eta}$ near $+\infty$. Then

$$t^\beta(-u'(t))^\alpha \leq c_1 \int_{t_0}^t r^{\sigma+\lambda(-\beta/\alpha+1)+\lambda\eta} dr = O(1) \quad \text{as } t \rightarrow \infty.$$

This contradicts the property of slowly decaying solution $\lim_{t \rightarrow \infty} t^\beta(-u'(t))^\alpha = \infty$. Hence Case (a) never occurs. As in the proof of [6, Theorem 1.3], we can show that Case (b) never occurs. Hence we have $\liminf_{s \rightarrow \infty} v(s) > 0$.

The remainder of the proof of the fact $\lim_{s \rightarrow \infty} v(s) = 1$ proceeds as in the proof of [6, Theorem 1.3]. We leave them to the reader.

Proof of Theorem 2. As in the proof of Theorem 1, we will show that $\lim_{s \rightarrow \infty} v(s) = 1$, where $v(s)$ is introduced in Lemma 2. Define

$$w = (kv - \dot{v})^\alpha. \quad (3.13)$$

By Eq (3.2) we know that

$$\dot{w} + \{\beta - \alpha(k+1)\}w - \hat{C}^{\lambda-\alpha}\{1 + \delta(s)\}v^\lambda = 0.$$

Let us rewrite this equation as

$$\dot{w} + aw - b\{1 + \delta(s)\}v^\lambda = 0, \quad (3.14)$$

where we have put $\beta - \alpha(k+1) = a$ and $\hat{C}^{\lambda-\alpha} = b$. We therefore find that

$$v = b^{-1/\lambda}(1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda},$$

and w satisfied the ODE

$$((1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda})' - k(1 + \delta(s))^{-1/\lambda}(\dot{w} + aw)^{1/\lambda} + b^{1/\lambda}w^{1/\alpha} = 0. \quad (3.15)$$

We note, by the definition (3.13), (3.14), and Lemma 2, that $w, \dot{w} = O(1)$ as $s \rightarrow \infty$. By putting $(1 + \delta(s))^{-1/\lambda} = h(s)$, $1/\lambda = \rho$, and $1/\alpha = \gamma$, we can rewrite (3.15) simply as

$$(h(s)(\dot{w} + aw)^\rho)' - kh(s)(\dot{w} + aw)^\rho + b^\rho w^{1/\alpha} = 0. \quad (3.16)$$

We note that our assumptions (2.4) are equivalent to

$$\lim_{s \rightarrow \infty} \dot{\delta}(s) = 0 \quad (3.17)$$

and

$$\int_0^\infty |\dot{\delta}(s)| ds < \infty. \quad (3.18)$$

It should be emphasized that Eq (3.16) is equivalent to

$$\ddot{w} + \left[a - \frac{k}{\rho} + \frac{\dot{h}(s)}{\rho h(s)} \right] \dot{w} + \frac{a}{\rho} \left[\frac{\dot{h}(s)}{h(s)} - k \right] w + \frac{b^\rho}{\rho h(s)} (\dot{w} + aw)^{1-\rho} w^\gamma = 0 \quad (3.19)$$

By using (3.18) and computing as in the proof of Theorem 1, we find from Eq (3.16) that

$$(a - k) \int_{s_0}^s h(r)(\dot{w} + aw)^\rho \dot{w} dr = O(1) \quad \text{as } s \rightarrow \infty. \quad (3.20)$$

Notice that the assumption $\beta - \alpha(k + 1) - k \neq 0$ means that $a - k \neq 0$. Since $\alpha \geq 1$ and $\lambda > \alpha$, we have $\rho < 1$. So inequality (3.8) implies, as before, that

$$\{(\dot{w} + aw)^\rho - (aw)^\rho\} \dot{w} \geq c_0 \dot{w}^2 \{|\dot{w} + aw| + |aw|\}^{\rho-1};$$

that is,

$$h(r)(\dot{w} + aw)^\rho \dot{w} \geq a^\rho h(r) w^\rho \dot{w} + c_1 h(r) \dot{w}^2$$

for some constant $c_1 > 0$. Hence by (3.20) and the fact that $h(\infty) = 1$, we find that

$$c_2 \int_{s_0}^s h(r) w^\rho \dot{w} dr + c_3 \int_{s_0}^s \dot{w}^2 dr = O(1) \quad \text{as } s \rightarrow +\infty.$$

By integral by parts and by using this relation, we find that $\int^\infty \dot{w}^2 ds < \infty$. Moreover, since $\rho < 1$, we find that $\lim_{s \rightarrow \infty} \dot{w}(s) = 0$ as in the proof of Theorem 1.

We want to show that $\liminf_{s \rightarrow \infty} w(s) > 0$. The proof is done by a contradiction. Firstly suppose that $w(s)$ decreases to 0 as $s \rightarrow \infty$. Then, as in the proof of Theorem 1, we know by [2, Chapter 3, Theorem 5] that for every $\eta > 0$

$$w(s) \leq e^{(-\beta + \alpha(k+1) + \eta)s} \quad \text{as } s \rightarrow \infty. \quad (3.21)$$

The definition (3.13) is equivalent to $(e^{-ks}v)' = -e^{-ks}w^{1/\alpha}$; and so

$$v(s) = e^{ks} \int_s^\infty e^{-kr} w^{1/\alpha} dr. \quad (3.22)$$

Here we have employed the fact that $\lim_{s \rightarrow \infty} v(s)/e^{ks} = 0$. Combining (3.21) with (3.22), we get the estimate $t^\beta |u'(t)| = O(1)$. Recall that this yields a contradiction.

Next, let $\liminf_{s \rightarrow \infty} w(s) = 0$ and \dot{w} change the sign in any neighborhood of $+\infty$. Define the auxiliary function $H(s)$ by

$$H(s) = k^\alpha \left[1 - \frac{\dot{h}(s)}{kh(s)} \right]^{\frac{\lambda\alpha}{\lambda-\alpha}}.$$

Then, in the region $0 < w < H(s)$, we have $\ddot{w} > 0$. On the other hand in the region $w > H(s)$, we have $\ddot{w} < 0$. Hence, we can find out two sequences $\{\xi_n\}$ and $\{\eta_n\}$ satisfying

$$\xi_n < \eta_n < \xi_{n+1} < \eta_{n+1} < \cdots; \lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} \eta_n = \infty;$$

and

$$w(\eta_n) \rightarrow 0, \quad w(\xi_n) = H(\xi_n) \rightarrow k^\alpha \quad \text{as } n \rightarrow \infty \quad \text{and} \quad \dot{w} \leq 0 \quad \text{on } [\xi_n, \eta_n]. \quad (3.23)$$

Multiplying (3.19) by \dot{w} and integrating the resulting equation on $[\xi_n, \eta_n]$, we have

$$\begin{aligned} & \frac{1}{2}(\dot{w}(\eta_n)^2 - \dot{w}(\xi_n)^2) + \left(a - \frac{k}{\rho}\right) \int_{\xi_n}^{\eta_n} \dot{w}^2 dr + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr \\ & - \frac{ak}{2\rho}(w(\eta_n)^2 - w(\xi_n)^2) + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr + \frac{b^\rho}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} (\dot{w} + aw)^{1-\rho} w^\gamma \dot{w} dr = 0. \end{aligned}$$

Noting the facts $\dot{w}(\infty) = 0$ and $\int^\infty \dot{w}^2 dr < \infty$, we have as $n \rightarrow \infty$

$$\begin{aligned} & o(1) + \frac{1}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr - \frac{ak}{2\rho}(o(1) - k^{2\alpha}) \\ & + \frac{a}{\rho} \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr + \frac{a^{1-\rho} b^\rho}{\rho} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho} \dot{w} dr \leq 0. \end{aligned} \quad (3.24)$$

Now, let us estimate each term of the above. We have firstly

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} \dot{w}^2 dr \right| \leq C_0 \sup_{[\xi_n, \infty)} |\dot{h}| \int_{\xi_n}^{\infty} \dot{w}^2 dr = o(1) \quad \text{as } n \rightarrow \infty;$$

and

$$\left| \int_{\xi_n}^{\eta_n} \frac{\dot{h}(r)}{h(r)} w \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{h(c_n)} \int_{\xi_n}^{\eta_n} w \dot{w} dr \right| = \left| \frac{\dot{h}(c_n)}{2h(c_n)} (w(\xi_n)^2 - w(\eta_n)^2) \right| = o(1) \quad \text{as } n \rightarrow \infty.$$

Here $C_0 > 0$ is a constant, and we have used a variant of the mean value theorem for integrals; that is c_n is a number satisfying $\xi_n < c_n < \eta_n$. Finally, we obtain

$$\begin{aligned} \int_{\xi_n}^{\eta_n} \frac{1}{h(r)} w^{1+\gamma-\rho} \dot{w} dr &= \int_{\xi_n}^{\eta_n} [h(r)^{-1} - 1] w^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2+\gamma-\rho} (w(\eta_n)^{2+\gamma-\rho} - w(\xi_n)^{2+\gamma-\rho}) \\ &= (h(d_n)^{-1} - 1) \int_{\xi_n}^{\eta_n} w^{1+\gamma-\rho} \dot{w} dr + \frac{1}{2+\gamma-\rho} (o(1) - k^{\alpha(2+\gamma-\rho)}) \\ &= o(1) - \frac{k^{\alpha(2+\gamma-\rho)}}{2+\gamma-\rho} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Here d_n is a number satisfying $\xi_n < d_n < \eta_n$. Therefore (3.24) can be simplified into

$$\frac{ak^{2\alpha+1}}{2\rho} + o(1) \leq \frac{a^{1-\rho} b^\rho k^{\alpha(2+\gamma-\rho)}}{\rho(2+\gamma-\rho)} \quad \text{as } n \rightarrow \infty.$$

This gives a contradiction. Hence we find that $\liminf_{s \rightarrow \infty} v(s) > 0$.

Arguing as in the proof of Theorem 1, we will show that $\lim_{s \rightarrow \infty} v(s) = 1$. The details are left to the reader.

To see Theorem 3, we will show that $\lim_{s \rightarrow \infty} v(s) = 1$, where $v(s)$ is introduced by Lemma 2, as before. However, we can not help omitting the proof for the lack of space.

References

- [1] M.-F. Bidaut-Veron, Local and global behavior of solutions of quasilinear equations of Emden-Fowler Type, Arch. Rat. Mech. Anal., 107 (1988), 293-324.
- [2] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, D. C. Heath and Company, 1965, Boston.
- [3] O. Došlý and P. Řehák, Half-linear Differential Equations, Elsevier, 2005, Amsterdam.
- [4] Y. Furusho, T. Kusano and A. Ogata, Symmetric positive entire solutions of second-order quasilinear degenerate elliptic equations, Arch. Rat. Mech. Anal., 127 (1994), 231-254.
- [5] M. Guedda and L. Veron, Local and global properties of solutions of quasilinear elliptic equations, J. Differential Equations, 76 (1988), 159-189.
- [6] K. Kamo and H. Usami, Asymptotic forms of weakly increasing positive solutions for quasilinear ordinary differential equations, Electron. J. Differential Equations, 2007(2007), No.126, 1-12.
- [7] T. Kusano, A. Ogata and H. Usami, Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations, Japan. J. Math., 19 (1993), 131-147.

Ken-ichi Kamo
kamo@sapmed.ac.jp

Hiroyuki Usami
usami@mis.hiroshima-u.ac.jp